Homework 6 - Sketch of Solutions

#1 a is not a based map. Set $\hat{a}(x) = \frac{a(x)}{a(1)} = \frac{-x}{-1} = x$

$\deg a = \deg \hat{a} = \deg(\mathrm{id}) = 1$

#4 $(a) \Rightarrow (c)$: $b \in B$ $\quad r(b - frb) = 0 \quad b - frb \in \operatorname{Ker} r$

$b = frb + k \in f(A) + \operatorname{Ker} r$. $\quad f(A) \cap \operatorname{Ker} r = 0$

$\therefore B = f(A) \oplus \operatorname{Ker} r$

$(c) \Rightarrow (a)$: Let $p : B = f(A) \oplus D \longrightarrow f(A)$ be the projection

$r = f^{-1} p : B \rightarrow A$.

$(b) \Rightarrow (c)$ $\quad b \in B$, $g(b - sgb) = 0$. $\quad b - sgb = f(a)$ some $a \in A$

$B = f(A) + s(C)$. $\quad f(A) \cap s(C) = 0 \quad \therefore B = f(A) \oplus s(C)$

$(c) \Rightarrow (b)$ $\quad B = f(A) \oplus D$. $\quad C \approx B / f(A) = D \subseteq B$. This defines

$s : C \rightarrow B$.

#6 Suppose P is free with basis $S \subseteq P$. $\forall s \in S$, $h(s) \in \operatorname{Im} g$

$h(s) = g(b_s)$ some $b_s \in B$ $\quad$ Set $f(s) = b_s$ and extend

f to a homo. $f : P \rightarrow B$. Then $gf(s) = g(b_s) = h(s)$. $\therefore gf = h$

(since they agree on S).

Now suppose P is projective. Then $\exists$ free (abelian) F and

epimorphism $g : F \rightarrow P$ (any group is a quotient of a free

group) Consider

$$\begin{tikzcd} & F \arrow[d, "g"] \\ P \arrow[ur, dashed, "f"] \arrow[r, "\mathrm{id}"] & P \end{tikzcd}$$

$\exists f$, $gf = \mathrm{id}$

$\therefore f$ is a mono.

$\therefore f(P) \subseteq F$ is free (subgroup of a free group is free). $\therefore P$

$\approx f(P)$ is free.

#7 $\xi : \tilde{H}_0(X) \rightarrow H_0(X)$ is $\xi : \frac{\operatorname{Ker} \varepsilon}{B_0(X)} \longrightarrow \frac{C_0(X)}{B_0(X)}$ defined by

$\xi(a + B_0(X)) = a + B_0(X)$, $a \in \operatorname{Ker} \varepsilon \subseteq C_0(X)$. $\quad \therefore \xi$ is a mono.

$\varepsilon_* : H_0(X) \rightarrow \mathbb{Z}$ defined by $\varepsilon_*(a + B_0(X)) = \varepsilon(a)$ is an epi.

$\varepsilon_* \xi(a + B_0(X)) = \varepsilon(a) = 0$ since $a \in \operatorname{Ker} \varepsilon$. $\therefore \operatorname{Im} \xi \subseteq \operatorname{Ker} \varepsilon_*$. Now

suppose $\varepsilon_*(c + B_0(X)) = 0$, $c \in C_0(X)$ $\quad \therefore \varepsilon(c) = 0$ so

$c \in \operatorname{Ker} \varepsilon$. $\therefore c + B_0(X) \in \operatorname{Im} \xi$ $\therefore \operatorname{Ker} \varepsilon_* = \operatorname{Im} \xi$.

#8 $x = c + B_0(X) \in H_0(X)$, $c \in C_0(X)$.

$f_*(x) = fc + B_0(Y)$ $\varepsilon'_*(fc + B_0 Y) = \varepsilon' fc$

$\therefore \varepsilon'_* f_*(x) = \varepsilon fc$ and $\varepsilon_*(x) = \varepsilon c$ But $\varepsilon' fc = \varepsilon c$ (check it for c = point).

#10 Send $1 \in \mathbb{Z}_4$ to $(2,1) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ (an element of order 4)

Let $H = \langle (2,1) \rangle$ the subgroup generated by $(2,1)$. Show the coset $(1,0) + H$ has order 4. $\therefore$ The given ses exists

#11 (a) Let $x = i_2 y$, $y \in \operatorname{Ker} f_2$

$$f_3(x) = f_3 i_2 y = j_2 f_2 y = 0 \quad \therefore i_2(\operatorname{Ker} f_2) \subseteq \operatorname{Ker} f_3$$

Conversely, $x \in \operatorname{Ker} f_3$ $f_4 i_3 x = j_3 f_3 x = 0$ $\therefore i_3 x = 0$ (f_4 mono)

$x = i_2 y$ some y.

$$j_2 f_2 y = f_3 i_2 y = f_3 x = 0$$

$$\therefore f_2 y = j_1 f_1 z \text{ some } z \quad (f_1 \text{ epi})$$

$$= f_2 i_1 z \qquad \text{so } y - i_1 z \in \operatorname{Ker} f_2$$

and $i_2(y - i_1 z) = i_2 y - i_2 i_1 z = i_2 y = x$

$\therefore x \in i_2(\operatorname{Ker} f_2)$.

(b) omitted

f_1 epi, f_2 mono, f_4 mono $\Longrightarrow$ f_3 mono

f_2 epi, f_4 epi, f_5 mono $\Longrightarrow$ f_3 epi